Pergamon

Computers Math. Applic. Vol. 29, No. 12, pp. 27–38, 1995

0898-1221/95 $9.50 + 0.00

0898-1221(95)00055-0

A Comparison of the Different Extensions of a Weak Formulation of an Approximate Riemann Solver for Supercritical Flows and Their Relationship to Existing Schemes

P. GLAISTER
Department of Mathematics, University of Reading
P.O. Box 220, Whiteknights, Reading RG6 2AX, U.K.

(Received and accepted October 1994)

Abstract—A weak formulation of the Roe linearised Riemann solver proposed recently is examined in the context of the equations governing steady, supercritical flows. Two possible alternatives present themselves, and it is shown that these are equivalent and, moreover, are equivalent to existing schemes.

Keywords—Weak formulation, Riemann solver, Supercritical flows.

1. INTRODUCTION

In a recent paper, Toumi [1] presented a weak formulation of Roe's approximate Riemann solver based on a definition of a nonconservative product. Toumi first identifies the Lipschitz continuous path connecting two states that leads to the Roe-averaged state [2] for an ideal gas, and then constructs a generalised Roe-averaged matrix for the Euler equations with real gases by using the same path. In a recent paper [3], it is shown that extending and then employing the ideas presented in [1] to the two-dimensional, unsteady, shallow water equations leads to a well-known approximate Riemann solver. In this paper, we examine the steady, supercritical case in which the solution procedure differs from the unsteady case. We show that there are two possible lines of attack to this problem, and subsequently, that these result in the same scheme; this scheme is currently available in the literature. This work is seen as an important first step to generalising the original weak formulation to compressible flows in both the steady and unsteady cases for real gases.

2. SHALLOW WATER FLOWS

The two-dimensional shallow water equations governing steady, supercritical flows can be written as

$$\underset{\sim}{f}_x + \underset{\sim}{g}_y = \underset{\sim}{0}, \tag{2.1}$$

where

$$\underset{\sim}{f}(\underset{\sim}{u}) = \left(\rho u, \frac{1}{2}\rho^2 + \rho u^2, \rho uv\right)^{\top}, \tag{2.2}$$

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

$$\underset{\sim}{g}(\underset{\sim}{u}) = \left(\rho v, \rho v u, \frac{1}{2}\rho^2 + \rho v^2\right)^{\mathsf{T}}, \quad \text{and} \tag{2.3}$$

$$\underset{\sim}{u} = (\rho, \rho u, \rho v)^{\mathsf{T}}, \tag{2.4}$$

denotes the vector of conserved variables. The quantities $(\rho, u, v) = (\rho, u, v)(x, y)$ represent the nondimensional height and the two velocity components at a general position (x, y) in space. (We have considered the case of a horizontal, rectangular, frictionless channel since the terms that arise when this is not the case merely produce a source which is handled separately.)

3. STRUCTURE

We begin by noting some of the relevant structure associated with the system of equation (2.1) which is essential for subsequent sections.

First, equation (2.1) can be written as

$$\underset{\sim}{f}_x + A\underset{\sim}{f}_y = \underset{\sim}{0}, \tag{3.1}$$

where the Jacobian

$$A = \frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{f}} = \frac{1}{u(u^2-\rho)}\begin{pmatrix} v(\rho+u^2) & -uv & u^2-\rho \\ 0 & 0 & u(u^2-\rho) \\ 2\rho(u^2+v^2) & -u(\rho+v^2) & 2v(u^2-\rho) \end{pmatrix}, \tag{3.2}$$

has eigenvalues, λ_i, given by

$$\lambda_{1,2,3} = \frac{uv \pm \rho\sqrt{F^2-1}}{u^2-\rho}, \frac{v}{u}, \tag{3.3a–c}$$

where the local Froude number, F, is defined by

$$F^2 = \frac{u^2+v^2}{\rho}, \tag{3.4}$$

and is assumed to satisfy $F > 1$. This condition is equivalent to the flow being supercritical everywhere and hence that the eigenvalues are real and distinct.

Second, the Jacobians

$$P = \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{u}} = \begin{pmatrix} 0 & 1 & 0 \\ \rho-u^2 & 2u & 0 \\ -uv & v & u \end{pmatrix}, \tag{3.5}$$

and

$$Q = \frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{u}} = \begin{pmatrix} 0 & 0 & 1 \\ -uv & v & u \\ \rho-v^2 & 0 & 2v \end{pmatrix}, \tag{3.6}$$

satisfy

$$A = QP^{-1}. \tag{3.7}$$

Moreover, equation (2.1) can be written as

$$P\underset{\sim}{u}_x + Q\underset{\sim}{u}_y = \underset{\sim}{0}, \quad \text{or} \tag{3.8}$$

$$\underset{\sim}{u}_x + \mathcal{A}\underset{\sim}{u}_y = \underset{\sim}{0}, \tag{3.9}$$

where

$$\mathcal{A} = P^{-1}Q = \frac{1}{u(u^2-\rho)}\begin{pmatrix} u^2 v & -uv & u^2 \\ 0 & 0 & u(u^2-\rho) \\ \rho(u^2+v^2-\rho) & -uv^2 & v(2u^2-\rho) \end{pmatrix}, \tag{3.10}$$

where

$$P^{-1} = \frac{1}{u(u^2-\rho)} \begin{pmatrix} 2u^2 & -u & 0 \\ u(u^2-\rho) & 0 & 0 \\ v(\rho+u^2) & -uv & u^2-\rho \end{pmatrix}, \tag{3.11}$$

and that $\mathcal{A}$ has the same eigenvalues as A. Note that the eigenvectors of A and $\mathcal{A}$, say $\underset{\sim}{r}$ and $\underset{\sim}{e}$, respectively, are related through

$$P\underset{\sim}{e} = \underset{\sim}{r}. \tag{3.12}$$

4. LINEARISED RIEMANN SOLVER—DIRECT FORMULATION

In this section, we review the direct formulation of the linearised Riemann solver approach as applied to the system equation (2.1), a fuller account of which can be found in [4].

It is well-known that solutions of equation (2.1) will exhibit oblique jumps in space, and for this reason, a numerical method that is capable of handling discontinuities (shocks) for time-dependent conservation laws is appropriate here. The scheme of Roe [2] for the Euler equations has been adapted to treat steady, supersonic flows, and in particular, steady, supercritical flows [4]. The construction of this scheme begins by defining the linearised Riemann problem for (2.1) corresponding to (3.1),

$$\underset{\sim}{f}_x + \tilde{A}(\underset{\sim}{u}_L, \underset{\sim}{u}_R)\underset{\sim}{f}_y = \underset{\sim}{0}, \tag{4.1a}$$

$$\underset{\sim}{u}(x_0, y) = \begin{cases} \underset{\sim}{u}_L & \text{if } y < 0, \\ \underset{\sim}{u}_R & \text{if } y > 0, \end{cases} \tag{4.1b}$$

where $\tilde{A}(\underset{\sim}{u}_L, \underset{\sim}{u}_R)$ is a constant matrix which depends on the data $(\underset{\sim}{u}_L, \underset{\sim}{u}_R)$ either side of the discontinuity at $y = 0$ along the line $x = x_0$. We are assuming that boundary conditions are known and the approximate solution is determined by searching in the x-direction, which is assumed to be aligned with the predominant flow direction and where the flow is wholly supercritical in this direction, i.e., $u^2(x,y) - \rho(x,y) > 0$ for all x, y.

The matrix $\tilde{A}$ is required to satisfy the jump-capturing property

$$\tilde{A}\Delta\underset{\sim}{f} = \Delta\underset{\sim}{g}, \tag{4.2}$$

for all jumps $\Delta\underset{\sim}{f}$. The approach of Roe is to introduce the parameter vector

$$\underset{\sim}{w} = (w_1, w_2, w_3)^{\mathsf{T}} = (\sqrt{\rho}, \sqrt{\rho}\,u, \sqrt{\rho}\,v)^{\mathsf{T}}, \tag{4.3}$$

then write

$$\Delta\underset{\sim}{f} = \tilde{R}\Delta\underset{\sim}{w}, \quad \text{and} \tag{4.4}$$

$$\Delta\underset{\sim}{g} = \tilde{S}\Delta\underset{\sim}{w}, \tag{4.5}$$

so that

$$\Delta\underset{\sim}{g} = \tilde{S}\tilde{R}^{-1}\Delta\underset{\sim}{f}, \quad \text{i.e.,} \tag{4.6}$$

$$\tilde{A} = \tilde{S}\,\tilde{R}^{-1}. \tag{4.7}$$

Expressing $\underset{\sim}{f}$ and $\underset{\sim}{g}$ in terms of $\underset{\sim}{w}$, we see that

$$\underset{\sim}{f}(\underset{\sim}{u}(\underset{\sim}{w})) = (w_1 w_2, w_1^4/2 + w_2^2, w_2 w_3)^{\mathsf{T}}, \quad \text{and} \tag{4.8}$$

$$\underset{\sim}{g}(\underset{\sim}{u}(\underset{\sim}{w})) = (w_1 w_3,\ w_2 w_3,\ w_1^4/2 + w_3^2)^{\mathsf{T}} \tag{4.9}$$

and in view of the identities

$$\Delta(ab) = \bar{a}\Delta b + \bar{b}\Delta a, \tag{4.10}$$
$$\Delta(a^2) = 2\bar{a}\Delta a, \tag{4.11}$$

and

$$\begin{aligned}\Delta(a^4) = a_R^4 - a_L^4 &= (a_R - a_L)\left(a_R^3 + a_R^2 a_L + a_R a_L^2 + a_L^3\right) \\ &= \left(a_R^3 + a_R^2 a_L + a_R a_L^2 + a_L^3\right)\Delta a,\end{aligned} \tag{4.12}$$

where the overbar $\bar{a} = (1/2)(a_L + a_R)$ denoted the arithmetic mean and $\Delta a = a_R - a_L$, then we have

$$\tilde{R} = \begin{pmatrix} \bar{w}_2 & \bar{w}_1 & 0 \\ 2\hat{w}_1^3 & 2\bar{w}_2 & 0 \\ 0 & \bar{w}_3 & \bar{w}_2 \end{pmatrix}, \quad \text{and} \tag{4.13}$$

$$\tilde{S} = \begin{pmatrix} \bar{w}_3 & 0 & \bar{w}_1 \\ 0 & \bar{w}_3 & \bar{w}_2 \\ 2\hat{w}_1^3 & 0 & 2\bar{w}_3 \end{pmatrix}, \tag{4.14}$$

where

$$\hat{w}_1^3 = \frac{1}{4}\left(w_{1_R}^3 + w_{1_R}^2 w_{1_L} + w_{1_R} w_{1_L}^2 + w_{1_L}^3\right). \tag{4.15}$$

Direct computation then shows that

$$\tilde{A} = \frac{1}{\tilde{u}(\tilde{u}^2 - \tilde{\rho})}\begin{pmatrix} \tilde{v}(\tilde{\rho} + \tilde{u}^2) & -\tilde{u}\tilde{v} & \tilde{u}(\tilde{u}^2 - \tilde{\rho}) \\ 0 & 0 & \tilde{u}(\tilde{u}^2 - \tilde{\rho}) \\ 2\tilde{\rho}(\tilde{u}^2 + \tilde{v}^2) & -\tilde{u}(\tilde{\rho} + \tilde{v}^2) & 2\tilde{v}(\tilde{u}^2 - \tilde{\rho}) \end{pmatrix}, \tag{4.16}$$

where we have set

$$\tilde{u} = \frac{\bar{w}_2}{\bar{w}_1} = \frac{\sqrt{\rho_L}u_L + \sqrt{\rho_R}u_R}{\sqrt{\rho_L} + \sqrt{\rho_R}}, \tag{4.17}$$

$$\tilde{v} = \frac{\bar{w}_3}{\bar{w}_1} = \frac{\sqrt{\rho_L}v_L + \sqrt{\rho_R}v_R}{\sqrt{\rho_L} + \sqrt{\rho_R}}, \tag{4.18}$$

and noted that

$$\hat{w}_1^3 = \frac{1}{4}\left(w_{1_L} + w_{1_R}\right)\left(w_{1_L}^2 + w_{1_R}^2\right) = \bar{w}_1\overline{w_1^2}, \tag{4.19}$$

giving rise to the definition

$$\tilde{\rho} = \overline{w_1^2} = \frac{1}{2}(\rho_L + \rho_R) = \bar{\rho}. \tag{4.20}$$

The matrix (4.16) is clearly an approximation to (3.2). The corresponding eigenvalues of $\tilde{A}$ are then

$$\tilde{\lambda}_{1,2,3} = \frac{\tilde{u}\tilde{v} \pm \tilde{\rho}\sqrt{\tilde{F}^2 - 1}}{\tilde{u}^2 - \tilde{\rho}}, \ \frac{\tilde{v}}{\tilde{u}}, \tag{4.21a–c}$$

where

$$\tilde{F}^2 = \frac{\tilde{u}^2 + \tilde{v}^2}{\tilde{\rho}}. \tag{4.22}$$

In order to solve (2.1) numerically, however, it is necessary to consider the corresponding linearised version of (3.9). To achieve this, we observe that by writing $\Delta\underset{\sim}{u}$ in terms of $\Delta\underset{\sim}{w}$ as

$$\Delta\underset{\sim}{u} = \tilde{T}\Delta\underset{\sim}{w}, \tag{4.23}$$

so that (4.4) and (4.5) become

$$\Delta \underset{\sim}{f} = \tilde{P}\Delta \underset{\sim}{u}, \quad \text{and} \tag{4.24}$$

$$\Delta \underset{\sim}{g} = \tilde{Q}\Delta \underset{\sim}{u}, \quad \text{where} \tag{4.25}$$

$$\tilde{P} = \tilde{R}\tilde{T}^{-1}, \quad \text{and} \tag{4.26}$$

$$\tilde{Q} = \tilde{S}\tilde{T}^{-1}, \tag{4.27}$$

represent approximations to the Jacobians P and Q given by (3.5) and (3.6). Determining $\tilde{T}$ we have, using (4.10) and (4.11),

$$\tilde{T} = \begin{pmatrix} 2\bar{w}_1 & 0 & 0 \\ \bar{w}_2 & \bar{w}_1 & 0 \\ \bar{w}_3 & 0 & \bar{w}_1 \end{pmatrix}, \quad \text{so that} \tag{4.28}$$

$$\tilde{P} = \begin{pmatrix} 0 & 1 & 0 \\ \tilde{\rho} - \tilde{u}^2 & 2\tilde{u} & 0 \\ -\tilde{u}\tilde{v} & \tilde{v} & \tilde{u} \end{pmatrix}, \quad \text{and} \tag{4.29}$$

$$\tilde{Q} = \begin{pmatrix} 0 & 0 & 1 \\ -\tilde{u}\tilde{v} & \tilde{v} & \tilde{u} \\ \tilde{\rho} - \tilde{v}^2 & 0 & 2\tilde{v} \end{pmatrix}, \tag{4.30}$$

where $\tilde{\rho}$, $\tilde{u}$ and $\tilde{v}$ are given by (4.17), (4.18) and (4.20). Having determined $\tilde{P}$ and $\tilde{Q}$, the corresponding linearised versions of (3.8) or (3.9) are

$$\tilde{P}\underset{\sim}{u}_x + \tilde{Q}\,\underset{\sim}{u}_y = \underset{\sim}{0}, \quad \text{or} \tag{4.31}$$

$$\underset{\sim}{u}_x + \tilde{\mathcal{A}}\,\underset{\sim}{u}_y = \underset{\sim}{0}, \tag{4.32}$$

where

$$\tilde{\mathcal{A}} = \tilde{P}^{-1}\tilde{Q} = \frac{1}{\tilde{u}(\tilde{u}^2 - \tilde{\rho})} \begin{pmatrix} \tilde{u}^2\tilde{v} & -\tilde{u}\tilde{v} & \tilde{u}^2 \\ 0 & 0 & \tilde{u}(\tilde{u}^2 - \tilde{\rho}) \\ \tilde{\rho}(\tilde{u}^2 + \tilde{v}^2 - \tilde{\rho}) & -\tilde{u}\tilde{v}^2 & \tilde{v}(2\tilde{u}^2 - \tilde{\rho}) \end{pmatrix}, \tag{4.33}$$

is an approximation to $\mathcal{A}$ in (3.10), and has the same eigenvalues as $\tilde{A}$. Note also that the approximate eigenvectors of $\tilde{A}$ and $\tilde{\mathcal{A}}$, denoted by $\underset{\sim}{\tilde{r}}$ and $\underset{\sim}{\tilde{e}}$, respectively, are related through

$$\tilde{P}\,\underset{\sim}{\tilde{e}} = \underset{\sim}{\tilde{r}}. \tag{4.34}$$

The numerical scheme presented in [4], whose associated structure is reviewed here, is based on upwind differencing applied to (4.33). In the next section, we consider the weak formulation of the linearised Riemann solver, and then, in the remaining sections propose two alternative approaches to this for the equations of steady, supercritical flows.

5. AN APPROXIMATE RIEMANN SOLVER (WEAK FORMULATION)

In [1], it is proposed solving a system of equations of the form

$$\underset{\sim}{q}_t + \underset{\sim}{h}(\underset{\sim}{q})_x = \underset{\sim}{0}, \tag{5.1}$$

also via locally linearised Riemann problems of the form

$$\underset{\sim}{q}_t + H(\underset{\sim}{q}_L, \underset{\sim}{q}_R)_\Phi\, \underset{\sim}{q}_x = \underset{\sim}{0}, \tag{5.2}$$

$$\underset{\sim}{q}(x,0) = \begin{cases} \underset{\sim}{q}_L & \text{if } x < 0, \\ \underset{\sim}{q}_R & \text{if } x > 0, \end{cases} \tag{5.3}$$

where $H(\underset{\sim}{u}_L, \underset{\sim}{u}_R)_{\underset{\sim}{\Phi}}$ is a constant matrix which depends on the data $(\underset{\sim}{q}_L, \underset{\sim}{q}_R)$ and on the path $\underset{\sim}{\Phi}(s; \underset{\sim}{q}_L, \underset{\sim}{q}_R)$, and satisfies

$$\int_0^1 H(\underset{\sim}{\Phi}(s; \underset{\sim}{q}_L, \underset{\sim}{q}_R)) \frac{\partial \underset{\sim}{\Phi}}{\partial s}(s; \underset{\sim}{q}_L, \underset{\sim}{q}_R)\, ds = H(\underset{\sim}{q}_L, \underset{\sim}{q}_R)_{\underset{\sim}{\Phi}} (\underset{\sim}{q}_R - \underset{\sim}{q}_L), \tag{5.4}$$

$$H(\underset{\sim}{q}, \underset{\sim}{q})_{\underset{\sim}{\Phi}} = H(\underset{\sim}{q}), \quad \text{and} \tag{5.5}$$

$$H(\underset{\sim}{q}_L, \underset{\sim}{q}_R)_{\underset{\sim}{\Phi}} \quad \text{has real eigenvalues and a complete set of eigenvectors,} \tag{5.6}$$

where

$$H = \frac{\partial \underset{\sim}{h}}{\partial \underset{\sim}{q}}, \tag{5.7}$$

is the Jacobian of $\underset{\sim}{h}$, and where (5.4) is equivalent to the condition $\underset{\sim}{h}(\underset{\sim}{q}_R) - \underset{\sim}{h}(\underset{\sim}{q}_L) = H(\underset{\sim}{q}_L, \underset{\sim}{q}_R)_{\underset{\sim}{\Phi}} (\underset{\sim}{q}_R - \underset{\sim}{q}_L)$.

The canonical path (a straight line) linking $\underset{\sim}{q}_L$ and $\underset{\sim}{q}_R$

$$\underset{\sim}{\Phi}(s; \underset{\sim}{q}_L, \underset{\sim}{q}_R) = \underset{\sim}{q}_L + s(\underset{\sim}{q}_R - \underset{\sim}{q}_L), \qquad s \in [0,1], \tag{5.8}$$

gives

$$H(\underset{\sim}{q}_L, \underset{\sim}{q}_R)_{\underset{\sim}{\Phi}} = \int_0^1 H\left(\underset{\sim}{q}_L + s(\underset{\sim}{q}_R - \underset{\sim}{q}_L)\right) ds. \tag{5.9}$$

The Riemann solver in [1] is constructed by letting $\underset{\sim}{f}_0$ be a smooth function such that

$$\underset{\sim}{f}_0(\underset{\sim}{w}_L) = \underset{\sim}{q}_L, \quad \underset{\sim}{f}_0(\underset{\sim}{w}_R) = \underset{\sim}{q}_R, \quad \text{and} \quad A_0(\underset{\sim}{w}) = \frac{\partial \underset{\sim}{f}_0}{\partial \underset{\sim}{w}}$$

is a regular matrix for every state $\underset{\sim}{w}$. The path chosen linking the two states $\underset{\sim}{q}_L$ and $\underset{\sim}{q}_R$ is then

$$\underset{\sim}{\Phi}_0(s; \underset{\sim}{q}_L, \underset{\sim}{q}_R) = \underset{\sim}{f}_0\left(\underset{\sim}{w}_L + s(\underset{\sim}{w}_R - \underset{\sim}{w}_L)\right), \tag{5.10}$$

and this leads to the matrix

$$H(\underset{\sim}{q}_L, \underset{\sim}{q}_R)_{\underset{\sim}{\Phi}_0} = C(\underset{\sim}{q}_L, \underset{\sim}{q}_R)_{\underset{\sim}{\Phi}_0} B(\underset{\sim}{q}_L, \underset{\sim}{q}_R)^{-1}_{\underset{\sim}{\Phi}_0}, \tag{5.11}$$

where

$$B(\underset{\sim}{q}_L, \underset{\sim}{q}_R)_{\underset{\sim}{\Phi}_0} = \int_0^1 A_0\left(\underset{\sim}{w}_L + s(\underset{\sim}{w}_R - \underset{\sim}{w}_L)\right) ds, \quad \text{and} \tag{5.12}$$

$$C(\underset{\sim}{q}_L, \underset{\sim}{q}_R)_{\underset{\sim}{\Phi}_0} = \int_0^1 H\left(\underset{\sim}{f}_0(\underset{\sim}{w}_L + s(\underset{\sim}{w}_R - \underset{\sim}{w}_L)\right) A_0\left(\underset{\sim}{w}_L + s(\underset{\sim}{w}_R - \underset{\sim}{w}_L)\right) ds, \tag{5.13}$$

which satisfies (5.4)–(5.7).

Our aim now is to look at two alternative ways of applying this Riemann solver to the equations of flow in Section 2, to compare them, and then to compare them with the scheme of Section 4.

6. APPLICATION TO STEADY, SUPERCRITICAL FLOWS

We now outline the two interpretations of the application of the scheme of Section 5 to the equations of Section 2, and then look at these in detail.

First Approach

In this approach, we consider the linearised version of (3.8)

$$P(\underset{\sim}{u}_L, \underset{\sim}{u}_R)_{\Phi_0}\, \underset{\sim}{u}_x + Q(\underset{\sim}{u}_L, \underset{\sim}{u}_R)_{\Phi_0}\, \underset{\sim}{u}_y = \underset{\sim}{0}, \tag{6.1}$$

where the matrices $\underset{\sim}{P}_{\Phi_0} = P(\underset{\sim}{u}_L, \underset{\sim}{u}_R)_{\Phi_0}$ and $\underset{\sim}{Q}_{\Phi_0} = Q(\underset{\sim}{u}_L, \underset{\sim}{u}_R)_{\Phi_0}$ are constructed, in turn, in the same way that $\underset{\sim}{H}(\underset{\sim}{q}_L, \underset{\sim}{q}_R)_{\Phi_0}$ is in Section 5. Essentially, this is like treating, in turn, the 'one-dimensional' operator split versions

$$\underset{\sim}{u}_t + \underset{\sim}{f}_x = \underset{\sim}{0}, \quad \text{and}$$
$$\underset{\sim}{u}_t + \underset{\sim}{g}_y = \underset{\sim}{0},$$

of the unsteady system

$$\underset{\sim}{u}_t + \underset{\sim}{f}_x + \underset{\sim}{g}_y = \underset{\sim}{0}.$$

Having achieved this, then the corresponding numerical scheme for (2.1) is obtained by rewriting (6.1) as

$$\underset{\sim}{u}_x + \underset{\sim}{\mathcal{A}}_{\Phi_0}\, \underset{\sim}{u}_y = \underset{\sim}{0}, \quad \text{where} \tag{6.2}$$

$$\underset{\sim}{\mathcal{A}}_{\Phi_0} = \underset{\sim}{P}_{\Phi_0}^{-1}\, \underset{\sim}{Q}_{\Phi_0}. \tag{6.3}$$

The associated linearised problem to (3.1) is then

$$\underset{\sim}{f}_x + \underset{\sim}{Q}_{\Phi_0}\, \underset{\sim}{P}_{\Phi_0}^{-1}\, \underset{\sim}{f}_y = \underset{\sim}{0}. \tag{6.4}$$

Second Approach

Here we consider directly the linearised problem associated with (3.1), viz.

$$\underset{\sim}{f}_x + A(\underset{\sim}{f}_L, \underset{\sim}{f}_R)_{\Phi_0}\, \underset{\sim}{f}_y, \tag{6.5}$$

where the matrix $\underset{\sim}{A}_{\Phi_0} = A(\underset{\sim}{f}_L, \underset{\sim}{f}_R)_{\Phi_0}$ is constructed as given in Section 5, and where the vector $\underset{\sim}{q}$ in Section 5 in this approach is the vector $\underset{\sim}{f}$, which is in contrast to that of the first approach where the vector $\underset{\sim}{q}$ is identified with $\underset{\sim}{u}$. Note also that since no approximations to P and Q are derived, then there is no natural approximation to $\mathcal{A}$ which is necessary for a numerical scheme based on a corresponding linearised problem for equation (3.9).

We now derive, in detail, the scheme resulting from each of the above two approaches.

7. FIRST APPROACH

We begin by constructing $\underset{\sim}{P}_{\Phi_0}$ by writing the parameter vector

$$\underset{\sim}{w} = (w_1, w_2, w_3)^{\top} = (\sqrt{\rho}, \sqrt{\rho}\, u, \sqrt{\rho}\, v)^{\top}, \tag{7.1}$$

so that

$$\underset{\sim}{f}_0(\underset{\sim}{w}) = \underset{\sim}{u} = (\rho, \rho u, \rho v)^{\top} = (w_1^2, w_1 w_2, w_1 w_3)^{\top}, \tag{7.2}$$

and hence,

$$A_0 = \frac{\partial \underset{\sim}{f}_0}{\partial \underset{\sim}{w}} = \begin{pmatrix} 2w_1 & 0 & 0 \\ w_2 & w_1 & 0 \\ w_3 & 0 & w_1 \end{pmatrix}. \tag{7.3}$$

From (5.12) and (7.3)

$$B_P(\underset{\sim}{u}_L, \underset{\sim}{u}_R)_{\underset{\sim}{\Phi}_0} = \int_0^1 A_0 \left(\underset{\sim}{w}_L + (s(\underset{\sim}{w}_R - \underset{\sim}{w}_L) \right) ds = \begin{pmatrix} 2\overline{w}_1 & 0 & 0 \\ \overline{w}_2 & \overline{w}_1 & 0 \\ \overline{w}_3 & 0 & \overline{w}_1 \end{pmatrix}, \tag{7.4}$$

where the overbar denotes the arithmetic mean of left and right states, $\overline{\underset{\sim}{w}} = (1/2)\left(\underset{\sim}{w}_L + \underset{\sim}{w}_R\right)$. To construct the matrix $C_P\left(\underset{\sim}{u}_L, \underset{\sim}{u}_R\right)_{\underset{\sim}{\Phi}_0}$ (having found $B_P\left(\underset{\sim}{u}_L, \underset{\sim}{u}_R\right)_{\underset{\sim}{\Phi}_0}$), and hence, $P(\underset{\sim}{u}_L, \underset{\sim}{u}_R)_{\underset{\sim}{\Phi}_0}$, it is necessary to write the Jacobian

$$P = \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{u}} = \begin{pmatrix} 0 & 1 & 0 \\ \rho - u^2 & 2u & 0 \\ -uv & v & u \end{pmatrix}, \tag{7.5}$$

as a function of $\underset{\sim}{w}$:

$$P\left(\underset{\sim}{u}(\underset{\sim}{w})\right) = \begin{pmatrix} 0 & 1 & 0 \\ w_1^2 - \dfrac{w_2^2}{w_1^2} & \dfrac{2w_2}{w_1} & 0 \\ -\dfrac{w_2 w_3}{w_1^2} & \dfrac{w_3}{w_1} & \dfrac{w_2}{w_1} \end{pmatrix}. \tag{7.6}$$

Combining (7.3) and (7.6) gives

$$P\left(\underset{\sim}{u}(\underset{\sim}{w})\right) A_0(\underset{\sim}{w}) = \begin{pmatrix} w_2 & w_1 & 0 \\ 2w_1^3 & 2w_2 & 0 \\ 0 & w_3 & w_2 \end{pmatrix}, \tag{7.7}$$

so that from (5.13)

$$\begin{aligned} C_P\left(\underset{\sim}{u}_L, \underset{\sim}{u}_R\right)_{\underset{\sim}{\Phi}_0} &= \int_0^1 P\left(\underset{\sim}{f}_0(\underset{\sim}{w}_L + s(\underset{\sim}{w}_R - \underset{\sim}{w}_L))\right) A_0\left(\underset{\sim}{w}_L + s(\underset{\sim}{w}_R - \underset{\sim}{w}_L)\right) ds \\ &= \begin{pmatrix} \overline{w}_2 & \overline{w}_1 & 0 \\ 2\widetilde{w_1^3} & 2\bar{w}_2 & 0 \\ 0 & \bar{w}_3 & \bar{w}_2 \end{pmatrix}, \end{aligned} \tag{7.8}$$

where again $\bar{\underset{\sim}{w}} = (1/2)\left(\underset{\sim}{w}_L + \underset{\sim}{w}_R\right)$ denotes the arithmetic mean, and $\widetilde{w_1^3}$ is an approximation to w_1^3 given by

$$\widetilde{w_1^3} = \int_0^1 (w_{1_L} + s(w_{1_R} - w_{1_L}))^3\, ds = \begin{cases} \dfrac{w_{1_R}^4 - w_{1_L}^4}{4(w_{1_R} - w_{1_L})}, & \text{if } w_{1_R} \neq w_{1_L}, \\ w_{1_L}^3 (= w_{1_R}^3), & \text{if } w_{1_R} = w_{1_L}. \end{cases} \tag{7.9a–b}$$

However, since

$$w_{1_R}^4 - w_{1_L}^4 = (w_{1_R} - w_{1_L})\left(w_{1_R}^3 + w_{1_R}^2 w_{1_L} + w_{1_R} w_{1_L}^2 + w_{1_L}^3\right), \tag{7.10}$$

then (7.9a) and (7.9b) become

$$\widetilde{w_1^3} = \frac{1}{4}\left(w_{1_R}^3 + w_{1_R}^2 w_{1_L} + w_{1_R} w_{1_L}^2 + w_{1_L}^3\right). \tag{7.11}$$

Combining (7.4), (7.8) and (7.11), we find that the matrix in (5.11) is

$$P_{\underset{\sim}{\Phi}_0} = P\left(\underset{\sim}{u}_L, \underset{\sim}{u}_R\right)_{\underset{\sim}{\Phi}_0} = C_P\left(\underset{\sim}{u}_L, \underset{\sim}{u}_R\right)_{\underset{\sim}{\Phi}_0} B_P\left(\underset{\sim}{u}_L, \underset{\sim}{u}_R\right)^{-1}_{\underset{\sim}{\Phi}_0} = \begin{pmatrix} 0 & 1 & 0 \\ \dfrac{\widetilde{w_1^3}}{\bar{w}_1} - \dfrac{\bar{w}_2^2}{\bar{w}_1^2} & \dfrac{2\bar{w}_2}{\bar{w}_1} & 0 \\ -\dfrac{\bar{w}_2\bar{w}_3}{\bar{w}_1^2} & \dfrac{\bar{w}_3}{\bar{w}_1} & \dfrac{\bar{w}_2}{\bar{w}_1} \end{pmatrix}. \tag{7.12}$$

Thus, since

$$\frac{\bar{w}_2}{\bar{w}_1} = \frac{\sqrt{\rho}_R u_R + \sqrt{\rho}_L u_L}{\sqrt{\rho}_R + \sqrt{\rho}_L} = \tilde{u}, \tag{7.13}$$

$$\frac{\bar{w}_3}{\bar{w}_1} = \frac{\sqrt{\rho}_R v_R + \sqrt{\rho}_L v_L}{\sqrt{\rho}_R + \sqrt{\rho}_L} = \tilde{v}, \quad \text{and} \tag{7.14}$$

$$\widetilde{w_1^3} = \frac{1}{4}\left(w_{1_R}^3 + w_{1_R}^2 w_{1_L} + w_{1_R} w_{1_L}^2 + w_{1_L}^3\right) = \frac{1}{4}\left(w_{1_R} + w_{1_L}\right)\left(w_{1_R}^2 + w_{1_L}^2\right) = \bar{w}_1 \overline{w_1^2}, \tag{7.15}$$

as in Section 4, then

$$P\left(\underset{\sim}{u}_L, \underset{\sim}{u}_R\right)_{\underset{\sim}{\Phi}_0} = \begin{pmatrix} 0 & 1 & 0 \\ \tilde{\rho} - \tilde{u}^2 & 2\tilde{u} & 0 \\ -\tilde{u}\tilde{v} & \tilde{v} & u \end{pmatrix}, \tag{7.16}$$

where

$$\tilde{\rho} = \overline{w_1^2} = \frac{1}{2}\left(w_{1_L}^2 + w_{1_R}^2\right) = \frac{1}{2}\left(\rho_L + \rho_R\right), \tag{7.17}$$

again denotes the arithmetic mean. The matrix $P_{\underset{\sim}{\Phi}_0}$ clearly represents an approximation to the Jacobian (3.5).

A similar calculation can be performed by replacing the matrix P by the matrix Q above, and this yields the approximate Jacobian to Q

$$Q_{\underset{\sim}{\Phi}_0} = Q\left(\underset{\sim}{u}_L, \underset{\sim}{u}_R\right)_{\underset{\sim}{\Phi}_0} = C_Q\left(\underset{\sim}{u}_L, \underset{\sim}{u}_R\right)_{\underset{\sim}{\Phi}_0} B_Q\left(\underset{\sim}{u}_L, \underset{\sim}{u}_R\right)^{-1}_{\underset{\sim}{\Phi}_0} = \begin{pmatrix} 0 & 0 & 1 \\ -\tilde{u}\tilde{v} & \tilde{v} & \tilde{u} \\ \tilde{\rho} - \tilde{v}^2 & 0 & 2\tilde{v} \end{pmatrix}. \tag{7.18}$$

The corresponding approximation in equation (6.5) is

$$\underset{\sim}{\mathcal{A}}_{\Phi_0} = \underset{\sim}{P}_{\Phi_0}^{-1} \underset{\sim}{Q}_{\Phi_0} = \frac{1}{\tilde{u}(\tilde{u}^2 - \tilde{\rho})} \begin{pmatrix} \tilde{u}^2\tilde{v} & -\tilde{u}\tilde{v} & \tilde{u}^2 \\ 0 & 0 & \tilde{u}(\tilde{u}^2 - \tilde{\rho}) \\ \tilde{\rho}(\tilde{u}^2 + \tilde{v}^2 - \tilde{\rho}) & -\tilde{u}\tilde{v}^2 & \tilde{v}(2\tilde{u}^2 - \tilde{\rho}) \end{pmatrix}. \tag{7.19}$$

Before making a comparison between the results here and those of Section 4, we consider the alternative approach outlined in Section 6.

8. SECOND APPROACH

For equation (2.1) with parameter vector

$$\underset{\sim}{w} = (w_1, w_2, w_3)^\top = (\sqrt{\rho}, \sqrt{\rho}\, u, \sqrt{\rho}\, v)^\top, \tag{8.1}$$

then the second approach of Section 6 sets

$$\underset{\sim}{f}_0(\underset{\sim}{w}) = \underset{\sim}{f} = \left(\rho u, \frac{1}{2}\rho^2 + \rho u^2, \rho u v\right)^\top = \left(w_1 w_2, w_1^4/2 + w_2^2, w_2 w_3\right)^\top, \tag{8.2}$$

so that in this case

$$A_0 = \frac{\partial \underset{\sim}{f}_0}{\partial \underset{\sim}{w}} = \begin{pmatrix} w_2 & w_1 & 0 \\ 2w_1^3 & 2w_2 & 0 \\ 0 & w_3 & w_2 \end{pmatrix}, \tag{8.3}$$

and hence, from (5.12) and (7.3)

$$B\left(\underset{\sim}{f}_L, \underset{\sim}{f}_R\right)_{\underset{\sim}{\Phi}_0} = \int_0^1 A_0\left(\underset{\sim}{w}_L + s(\underset{\sim}{w}_R - \underset{\sim}{w}_L)\right) ds = \begin{pmatrix} \overline{w}_2 & \bar{w}_1 & 0 \\ 2\widetilde{w_1^3} & 2\overline{w}_2 & 0 \\ 0 & \bar{w}_3 & \overline{w}_2 \end{pmatrix}, \tag{8.4}$$

where all quantities are as defined in Section 7.

Now, writing the Jacobian in (3.2)

$$A = \frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{f}} = \frac{1}{u(u^2-\rho)} \begin{pmatrix} v(\rho+u^2) & -uv & u^2-\rho \\ 0 & 0 & u(u^2-\rho) \\ 2\rho(u^2+v^2) & -u(\rho+v^2) & 2v(u^2-\rho) \end{pmatrix}, \tag{8.5}$$

as a function of $\underset{\sim}{w}$:

$$A\left(\underset{\sim}{f}(\underset{\sim}{w})\right) = \begin{pmatrix} \dfrac{w_3(w_1^4+w_2^2)}{w_2(w_2^2-w_1^4)} & \dfrac{-w_3w_1}{w_2^2-w_1^4} & \dfrac{w_1}{w_2} \\ 0 & 0 & 1 \\ \dfrac{2w_1^3(w_2^2+w_3^2)}{w_2(w_2^2-w_1^4)} & \dfrac{-(w_1^4+w_3^2)}{w_2^2-w_1^4} & \dfrac{2w_3}{w^2} \end{pmatrix}, \tag{8.6}$$

and then combining (8.3) and (8.6) gives

$$A\left(\underset{\sim}{f}(\underset{\sim}{w})\right) A_0(\underset{\sim}{w}) = \begin{pmatrix} w_3 & 0 & w_1 \\ 0 & w_3 & w_2 \\ 2w_1^3 & 0 & 2w_3 \end{pmatrix}, \tag{8.7}$$

so that from (5.13)

$$\begin{aligned} C(\underset{\sim}{f}_L, \underset{\sim}{f}_R)_{\underset{\sim}{\Phi}_0} &= \int_0^1 A\left(\underset{\sim}{f}_0(\underset{\sim}{w}_L + s(\underset{\sim}{w}_R - \underset{\sim}{w}_L))\right) A_0\left(\underset{\sim}{w}_L + s(\underset{\sim}{w}_R - \underset{\sim}{w}_L)\right) ds \\ &= \begin{pmatrix} \bar{w}_3 & 0 & \bar{w}_1 \\ 0 & \bar{w}_3 & \bar{w}_2 \\ 2\widetilde{w_1^3} & 0 & 2\bar{w}_3 \end{pmatrix}. \end{aligned} \tag{8.8}$$

Combining (8.4) and (8.8), we find that the matrix in (5.11) for (6.5) is

$$A(\underset{\sim}{f}_L, \underset{\sim}{f}_R)_{\Phi_0} = C(\underset{\sim}{f}_L, \underset{\sim}{f}_R)_{\Phi_0} B(\underset{\sim}{f}_L, \underset{\sim}{f}_R)_{\underset{\sim}{\Phi}_0}^{-1} = \begin{pmatrix} \dfrac{\bar{w}_3(\bar{w}_2^2+\bar{w}_1\bar{w}_1^3)}{\bar{w}_2(\bar{w}_2^2-\bar{w}_1\widetilde{w_1^3})} & \dfrac{\bar{w}_1\bar{w}_3}{\bar{w}_1\widetilde{w_1^3}-\bar{w}_2^2} & \dfrac{\bar{w}_1}{\bar{w}_2} \\ 0 & 0 & 1 \\ \dfrac{2(\bar{w}_2^2+\bar{w}_3^2)\widetilde{w_1^3}}{\bar{w}_2(\bar{w}_2^2-\bar{w}_1\widetilde{w_1^3})} & \dfrac{\bar{w}_1\widetilde{w_1^3}+\bar{w}_3^2}{\bar{w}_1\widetilde{w_1^3}-\bar{w}_2^2} & \dfrac{2\bar{w}_3}{\bar{w}_2} \end{pmatrix}, \tag{8.9}$$

and rewriting in terms of the averages $\tilde{u}$, $\tilde{v}$, and $\tilde{\rho}$ defined in Section 7, we have

$$A(\underset{\sim}{u}_L, \underset{\sim}{u}_R)_{\underset{\sim}{\Phi}_0} = \frac{1}{\tilde{u}(\tilde{u}^2-\tilde{\rho})} \begin{pmatrix} \tilde{v}(\tilde{\rho}+\tilde{u}^2) & -\tilde{u}\tilde{v} & \tilde{u}^2-\tilde{\rho} \\ 0 & 0 & \tilde{u}(\tilde{u}^2-\tilde{\rho}) \\ 2\tilde{\rho}(\tilde{u}^2+\tilde{v}^2) & -\tilde{u}(\tilde{\rho}+\tilde{v}^2) & 2\tilde{v}(\tilde{u}^2-\tilde{\rho}) \end{pmatrix}, \tag{8.10}$$

which clearly represents an approximation to the Jacobian (3.2).

In the next section, we compare the results obtained in this section, Sections 7 and 4.

9. A COMPARISON OF THE DIFFERENT FORMULATIONS

First, we observe that the matrices $\underset{\sim}{P}_{\Phi_0}$ and $\underset{\sim}{Q}_{\Phi_0}$ determined in Section 7 yield a linearised problem as noted in equation (6.4) with associated matrix $\underset{\sim}{Q}_{\Phi_0}\,\underset{\sim}{P}_{\Phi_0}^{-1}$, and evaluating this yields

$$\underset{\sim}{Q}_{\Phi_0}\underset{\sim}{P}_{\Phi_0}^{-1} = \frac{1}{\tilde{u}(\tilde{u}^2-\tilde{\rho})}\begin{pmatrix} \tilde{v}(\tilde{\rho}+\tilde{u}^2) & -\tilde{u}\tilde{v} & \tilde{u}^2-\tilde{\rho} \\ 0 & 0 & \tilde{u}(\tilde{u}^2-\tilde{\rho}) \\ 2\tilde{\rho}(\tilde{u}^2+\tilde{v}^2) & -\tilde{u}(\tilde{\rho}+\tilde{v}^2) & 2\tilde{v}(\tilde{u}^2-\tilde{\rho}) \end{pmatrix}, \tag{9.1}$$

which is precisely the matrix $\underset{\sim}{A}_{\Phi_0}$ determined in Section 8 using the alternative formulation, i.e.,

$$A(\underset{\sim}{f}_L,\underset{\sim}{f}_R)_{\Phi_0} = Q(\underset{\sim}{u}_L,\underset{\sim}{u}_R)_{\Phi_0}\,P(\underset{\sim}{u}_L,\underset{\sim}{u}_R)_{\Phi_0}^{-1}, \tag{9.2}$$

so that the two formulations of Sections 7 and 8 are equivalent. This result will be of importance when examining steady, supersonic, compressible flows, particularly since the construction of $\underset{\sim}{A}_{\Phi_0}$ using the second approach of Section 8 is the most straightforward. We again note, however, that the second approach of Section 8 does not yield intermediate matrices $\underset{\sim}{P}_{\Phi_0}$ and $\underset{\sim}{Q}_{\Phi_0}$, and thus no natural matrix $\underset{\sim}{\mathcal{A}}_{\Phi_0}$.

Second, having established the equivalence above, a direct comparison of the matrices $\tilde{P}$, $\tilde{Q}$, $\tilde{A}$ and $\tilde{\mathcal{A}}$ in Section 4 with the matrices $\underset{\sim}{P}_{\Phi_0}$, $\underset{\sim}{Q}_{\Phi_0}$, $\underset{\sim}{A}_{\Phi_0}$ and $\underset{\sim}{\mathcal{A}}_{\Phi_0}$, of Sections 7 and 8, shows that these matrices are, respectively, equivalent, and in particular, the two averages $\hat{w}_1^3$ in equation (4.15) and $\widetilde{w_1^3}$ in equation (7.11) are identical. Moreover, the eigenvalues of $\underset{\sim}{A}_{\Phi_0}$ and $\underset{\sim}{\mathcal{A}}_{\Phi_0}$ are those of the matrices $\tilde{A}$ and $\tilde{\mathcal{A}}$ as given by equation (4.21a–c) and (4.22).

Finally, we observe that the following additional relationships between the matrices of the various formulations hold

$$\tilde{R} = C_P(\underset{\sim}{u}_L,\underset{\sim}{u}_R)_{\Phi_0} = B(\underset{\sim}{f}_L,\underset{\sim}{f}_R)_{\Phi_0}, \tag{9.3}$$

$$\tilde{S} = C_Q(\underset{\sim}{u}_L,\underset{\sim}{u}_R)_{\Phi_0} = C(\underset{\sim}{f}_L,\underset{\sim}{f}_R)_{\Phi_0}, \quad \text{and} \tag{9.4}$$

$$\tilde{T} = B_P(\underset{\sim}{u}_L,\underset{\sim}{u}_R)_{\Phi_0} = B_Q(\underset{\sim}{u}_L,\underset{\sim}{u}_R)_{\Phi_0}, \tag{9.5}$$

so that

$$\underset{\sim}{Q}_{\Phi_0}\,\underset{\sim}{P}_{\Phi_0}^{-1} = (C_Q\,B_Q^{-1})(C_P\,B_P^{-1})^{-1} = C_Q\,B_Q^{-1}\,B_P\,C_P^{-1} = C_Q C_P^{-1} = \underset{\sim}{C}_{\Phi_0}\,\underset{\sim}{B}_{\Phi_0}^{-1} = \underset{\sim}{A}_{\Phi_0}; \tag{9.6}$$

hence,

$$\tilde{A} = \tilde{S}\,\tilde{R}^{-1} = \underset{\sim}{C}_{\Phi_0}\,\underset{\sim}{B}_{\Phi_0}^{-1} = \underset{\sim}{A}_{\Phi_0}, \tag{9.7}$$

and

$$\begin{aligned}\tilde{\mathcal{A}} = \tilde{P}^{-1}\tilde{Q} &= (\tilde{R}\tilde{T}^{-1})^{-1}(\tilde{S}\tilde{T}^{-1}) = \tilde{T}\,\tilde{R}^{-1}\,\tilde{S}\,\tilde{T}^{-1} = B_P\,C_P^{-1}\,C_Q\,B_Q^{-1} \\ &= (C_P\,B_P^{-1})^{-1}(C_Q\,B_Q^{-1}) = \underset{\sim}{P}_{\Phi_0}^{-1}\,\underset{\sim}{Q}_{\Phi_0} = \underset{\sim}{\mathcal{A}}_{\Phi_0}.\end{aligned} \tag{9.8}$$

10. CONCLUSIONS

We have considered two approaches to the weak formulation of a linearised Riemann solver for steady, supercritical, free-surface flows, and demonstrated that these are equivalent and are also equivalent to the direct formulation. It is intended to extend this work to steady, supersonic, compressible flows where the equivalence of these approaches can be utilised.

REFERENCES

1. I. Toumi, A weak formulation of Roe's approximate Riemann solver, *J. Comput. Phys.* **102** (2), 360–373 (1992).
2. P.L. Roe, Approximate Riemann solvers, parameter vectors and difference schemes, *J. Comput. Phys.* **43** (2), 357–372 (1981).
3. P. Glaister, An extension of Toumi's method and its application to the two-dimensional, unsteady, shallow water equations, *Mathl. Comput. Modelling* **21** (3), 93–98 (1995).
4. P. Glaister, A finite difference scheme for steady, supersonic, two-dimensional, compressible flow of real gases, *Computers Math. Applic.* **24** (4), 49–59 (1992).